On the Number of Idempotent Linear Transformations*

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ABSTRACT

Let M and N be two subspaces of a finite dimensional vector space V over a finite field F. We can count the number of all idempotent linear transformations T of V such that $R(T) \subset M$ and $N \subset N(T)$, where $R(T)$ and $N(T)$ denote the range space and the null space of T, respectively.

1. INTRODUCTION

(A) The number of all idempotent linear transformations of a finite dimensional vector space V over a finite field F with q elements has been considered by J. H. Hodges [4, 5], but further investigation of this problem has received very little attention. We note the following [6, Theorem A(vi)]:

(B) Let M and N be subspaces of a vector space V such that $\dim(V/N) = \dim M$. Let H be the set of all linear transformations T of V such that $R(T) = M$ and $N(T) = N$. Then H contains an idempotent iff M and N are complementary in V.

We remark that an idempotent linear transformation is uniquely determined by its range and kernel (by [2, Theorem –(ii) on p. 74]). If we replace M by V and N by (0) in (B), then (the first part of) (A) is equivalent to the number of all idempotent linear transformations T of V such that $R(T) \subset V$ and $(0) \subset N(T)$. (This is considered by J. H. Hodges [4, 5]).

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Therefore we can easily see that (A) and (B) are two extreme cases of that a count of the number of all idempotent linear transformations T of a finite dimensional vector space V over a finite field F such that $R(T) \subset M$ and $N \subset N(T)$, where M and N are two fixed subspaces of V, and this is a great generalization of (A) and (B) both.

2. A THEOREM AND LEMMAS

First we quote here the following known theorem [6, Theorem A(vi)] and we shall use this basic theorem.

THEOREM 1. *If $L(V)$ is the multiplicative semigroup of all linear transformations of a finite dimensional vector space V over a field F. Let N and M be subspaces of V such that* $\dim(V/N) = \dim M$. *Let H be the set of all elements T of $L(V)$ with $R(T) = M$ and $N(T) = N$. Then H contains an idempotent iff N and M are complementary in V, and this idempotent is the projection of V upon M which annuls N. If this is the case, H induces and is isomorphic with the full linear group $GL(M)$ on M, consisting of all nonsingular linear transformations of M.*

Let F be a finite field with q elements. Throughout this paper, let n be a fixed positive integer greater than 1. Let V be a vector space over F of dimension n. $\pi(V)$ denotes the collection of all subspaces of V and $\pi^n{}_r(V) = \pi_r(V)$ denotes the collection of all subspaces of V of dimension r $(0 \leqslant r \leqslant n)$. The following lemma is well known.

LEMMA 1. *Let $M, N \in \pi(V)$. Then* $\dim(M + N) + \dim(M \cap N) = \dim M + \dim N$.

NOTATION. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for M in $\pi_r(V)$. It will be convenient to write $M = [e_1, e_2, \ldots, e_r]$ to indicate that $\{e_1, e_2, \ldots, e_r\}$ is a basis for M.

LEMMA 2. *Let r and s be positive integers.*

(1) *If $M = [x_1, x_2, \ldots, x_r] \in \pi_r(V)$, $N = [y_1, y_2, \ldots, y_s] \in \pi_s(V)$ and $M \cap N = (0)$, then $\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s\}$ is a set of $(r + s)$ linearly independent vectors.*

(2) *Let $M \in \pi_r(V)$ and $N \in \pi_{n-r+s}(V)$, then $M \cap N \neq (0)$.*

The proof of Lemma 2 is trivial and we omit it. $|A|$ denotes the cardinality of a set A.

LEMMA 3. $|\pi_r(V)| = \begin{bmatrix} n \\ r \end{bmatrix}$ *is given by*

$$\frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-r+1} - 1)}{(q^r - 1)(q^{r-1} - 1) \cdots (q - 1)} = \begin{bmatrix} n \\ r \end{bmatrix} \quad \textit{for} \quad r \geqslant 1.$$

We define

$$\begin{bmatrix} r \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} r \\ s \end{bmatrix} = 0$$

for $s > r$. For the proof of Lemma 3 see [6, Theorem B].

NOTATION. (1) Let $M, N \in \pi(V)$. $[N, M]$ denotes a pair of M and N with $M \oplus N = V$, the direct sum of M and N.

(2) $D_r[N, M] = \{[N', M']\colon \ M' \subset M, \ \ N \subset N' \ \text{ and } \ \dim M' = r\}$, where N, M, N' and M' are members of $\pi(V)$.

(3) $\emptyset$ denotes the empty set.

LEMMA 4. *Let* $M, N \in \pi(V)$. *Then* $|D_0[N, M]| = 1$.

Proof. $D_0[N, M]$ contains just one element $[V, (0)]$.

LEMMA 5. *Let r and k be positive integers such that $1 \leqslant k \leqslant r$. Let $M \in \pi_r(V)$ and $N \in \pi_{n-r+k}(V)$. Then*

(1) $|D_i[N, M]| = 0$ *if* $r - k + 1 \leqslant i$.

(2) *Let* $M + N = V$. *Then*

$$|D_{r-k-t}[N, M]| = \begin{bmatrix} r - k \\ r - k - t \end{bmatrix} q^{(k+t)(r-k-t)},$$

for $t = 0, 1, 2, \ldots, r - k - 1$.

(3) *Let* $M + N \neq V$. *Then* $\dim(M \cap N) = k + m > k$, *and*

$$|D_{r-k-t}[N, M]|$$

$$= \begin{cases} \begin{bmatrix} r - k - m \\ r - k - t \end{bmatrix} q^{(r-k-t)(t+k+m)}, & \textit{if } m \leqslant t \ (t = m, m + 1, \ldots, r - k - 1), \\ 0, & \textit{if } m > t. \end{cases}$$

Proof. (1) If $[N', M'] \in D_i[N, M]$, then $M' \subset M$, $N \subset N'$, $\dim M' = i$ and $N' \oplus M' = V$. Then $n = \dim N' + \dim M' \geqslant \dim N + \dim M' = n - r + k + i \geqslant n - r + k + r - k + 1 = n + 1$, which is impossible.

(2) The proof consists of several steps. We shall show that the number m of all M_1 such that $[N_1, M_1] \in D_{r-k-t}[N, M]$ for a fixed N_1 is equal to $m_1 m_2$, where

$$m_1 = \begin{bmatrix} r-k \\ r-k-t \end{bmatrix} \quad \text{and} \quad m_2 = q^{k(r-k-t)}.$$

Then we shall show that the number m_3 of all N_1 such that $[N_1, M_1] \in D_{r-k-t}[N, M]$, for a fixed M_1, is equal to $q^{t(n-k-t)} = m_3$, and hence $|D_{r-k-t}[N, M]| = m_1 m_2 m_3$.

(i) We can see that $D_{r-k-t}[N, M] \neq \emptyset$. We can see, by Lemma 1, that $\dim(M \cap N) = k$ when $M + N = V$. Let $[N_1, M_1]$ be a member of $D_{r-k-t}[N, M]$. Then $M_1 \subset M$, $N \subset N_1$, $\dim M_1 = r - k - t$ and $M_1 \oplus N_1 = V$. Note that $M_1 \cap N \subset M_1 \cap N_1 = (0)$ and hence $M_1 \cap N = (0)$. Now let

$$M \cap N = [v_1, v_2, \ldots, v_k], \qquad M = [v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{r-k}]$$

and

$$N = [v_1, v_2, \ldots, v_k, u_{k+1}, u_{k+2}, \ldots, u_{n-r+k}].$$

Let $M_1 = [z_1, z_2, \ldots, z_{r-k-t}]$. Since M_1 is a subspace of M, z_i takes the form

$$z_i = \sum_{j=1}^{r-k} a_{ij} w_j + \sum_{j=1}^{k} b_{ij} v_j \qquad (i = 1, 2, \ldots, r-k-t), \qquad (a_{ij}, b_{ij} \in F).$$

Letting $A = (a_{ij})$ and $B = (b_{ip})$ $(i = 1, 2, \ldots, r-k-t; j = 1, 2, \ldots, r-k; p = 1, 2, \ldots, k)$, $C = (A \quad B)$ is a $(r-k-t) \times r$ matrix of rank $(r-k-t)$. Let $F(n_1, n_2)$ be the set of all $n_1 \times n_2$ matrices over F. Now we can say that each M_1 of $[N_1, M_1] \in D_{r-k-t}[N, M]$ corresponds to $C = (A \quad B) \in F(r-k-t, r)$ of rank $r-k-t$, for a fixed N_1, where $A \in F(r-k-t, r-k)$ and $B \in F(r-k-t, k)$, and we may write that $M_1 = M_{AB} \to C = (A \quad B)$.

(ii) We shall show that the rank of the matrix $A = (a_{ij})$ is equal to $(r-k-t)$. By elementary row operations on C, C can be transformed into a row reduced echelon matrix $D = E_s E_{s-1} \cdots E_2 E_1 C$, where E_i $(i = 1, 2, \ldots, s)$ are elementary matrices. Suppose that the rank of A is less than $r - k - t$. Then there is i in $\{r - k, r - k + 1, \ldots, r\}$ such that the ith column of D contains just one nonzero entry, say (i, j) entry of D is 1. Letting

$$G = (w_1 \quad w_2 \quad \cdots \quad w_{r-k} \quad v_1 \quad v_2 \quad \cdots \quad v_k),$$

a $1 \times r$ matrix, the jth row vector of DG^t is of the form

$$v_j + \sum_{s=j+1}^{k} c_s v_s = v \qquad (c_s \in F),$$

where G^t denotes the transposed matrix of G. We can see that $M_1 \cap N$ contains a nonzero vector v, contradicts the assumption that $M_1 \cap N = (0)$. Thus the rank of A is equal to $r - k - t$. Now we can say that each $M_{AB} = M_1$ of $[N_1, M_{AB}] \in D_{r-k-t}[N, M]$ corresponds to $C = (A \quad B)$ in $F(r - k - t, r)$ with $\operatorname{rank}(A) = r - k - t$, where

$$M_{AB} = [z_1, z_2, \ldots, z_{r-k-t}],$$

$$z_i = \sum_{j=1}^{r-k} a_{ij} w_j + \sum_{j=1}^{k} b_{ij} v_j,$$

$$A = (a_{ij}) \in F(r - k - t, r - k)$$

and

$$B = (b_{ij}) \in F(r - k - t, k).$$

(iii) Let

$$S_1 = \{A \in F(r - k - t, r - k) : \operatorname{rank}(A) = r - k - t\}$$

and let $A \in S_1$. Let $B, B' \in F(r - k - t, k)$.

We now show that for a fixed N_1, if $[N_1, M_{AB}], [N_1, M_{AB'}] \in D_{r-k-t}[N, M]$ and if $B \neq B'$, then $M_{AB} \neq M_{AB'}$. To show this let $B = (b_{ij})$ and $B' = (b'_{ij})$ be two distinct elements of $F(r - k - t, k)$. Let

$$y_i = \sum_{j=1}^{r-k} a_{ij} w_j + \sum_{j=1}^{k} b_{ij} v_j \quad \text{and} \quad y_i' = \sum_{j=1}^{r-k} a_{ij} w_j + \sum_{j=1}^{k} b'_{ij} v_j$$

$$(i = 1, 2, \ldots, r - k - t).$$

We shall show that $M_{AB} = [y_1, y_2, \ldots, y_{r-k-t}]$ and $M_{AB'} = [y_1', y_2', \ldots, y'_{r-k-t}]$ are two distinct vector spaces of dimension $(r - k - t)$. We suppose, to the contrary, that $M_{AB} = M_{AB'}$. For each i, we have that $y_i \in [y_1', y_2', \ldots, y'_{r-k-t}]$, and y_i takes the form $y_i = c_1 y_1' + c_2 y_2' + \cdots + c_{r-k-t} y'_{r-k-t}$. By computation, we can see that $c_1 = c_2 = \cdots = c_{i-1} = c_{i+1} = \cdots = c_{r-k-t} = 0$ and $c_i = 1$. Thus $y_i = y_i'$. We can obtain that $b_{ij} = b'_{ij}$ and hence $B = B'$. This contradiction proves that M_{AB} and $M_{AB'}$ are two distinct vector spaces.

Now it is not very difficult to see that there is a one-to-one correspondence

$$f\colon S_2 = \{M_{AB}\colon [N_1, M_{AB}] \in D_{r-k-t}[N, M], A \in S_1, B \in F(r-k-t, k) \text{ and } N_1 \text{ is fixed}\}$$

$$\to S_3 = \{C = (A \quad B) \in F(r-k-t, r)\colon A \in S_1\}.$$

Thus $m = |S_3|$ and $m = m_1 m_2$, where $m_1 = |S_1|$ and $m_2 = |F(r-k-t, k)| = q^{k(r-k-t)}$, the number of all $(r-k-t) \times k$ matrices B over F. Each $A = (a_{ij})$ in S_1 corresponds to $C = (A \quad 0)$ in S_3 and $M_{A0} = [x_1, x_2, \ldots, x_{r-k-t}]$, a subspace of M of dimension $r-k-t$, where $x_i = \sum_{j=1}^{r-k} a_{ij} w_j$. We recall that

$$M \cap N = [v_1, v_2, \ldots, v_k] \quad \text{and} \quad M = [v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_{r-k}].$$

Therefore $m_1 = |S_1|$ is equal to the number of all subspaces $[x_1, x_2, \ldots, x_{r-k-t}] = M_{A0}$ of the space $Y = [w_1, w_2, \ldots, w_{r-k}]$ of dimension $r-k-t$, that is,

$$|\pi^{r-k}_{r-k-t}(Y)| = \begin{bmatrix} r-k \\ r-k-t \end{bmatrix}$$

by Lemma 3. Thus

$$m = m_1 m_2 = \begin{bmatrix} r-k \\ r-k-t \end{bmatrix} q^{k(r-k-t)}.$$

(iv) We consider the number m_3 of all N' such that $[N', M'] \in D_{r-k-t}[N, M]$ for a fixed M'. Let

$$M' = M_{AB} = [y_1, y_2, \ldots, y_{r-k-t}], \quad y_i = x_i + \sum_{j=1}^{k} b_{ij} v_j, \quad x_i = \sum_{j=1}^{r-k} a_{ij} w_j$$

and $A = (a_{ij}) \in S_1$. We can find vectors $z_1, z_2, \ldots, z_t$ such that

$$M = [v_1, v_2, \ldots, v_k, x_1, x_2, \ldots, x_{r-k-t}, z_1, z_2, \ldots, z_t].$$

Letting

$$z_i' = z_i + \sum_{j=1}^{r-k-t} h_{ij} x_j \qquad (h_{ij} \in F),$$

we can prove that

$$N_H = [v_1, v_2, \ldots, v_k, u_{k+1}, u_{k+2}, \ldots, u_{n-r+k}, z_1', z_2', \ldots, z_t']$$

is a vector space of dimension $(n - r + k + t)$ such that $N_H + M_{AB} = V$ and $N_H \cap M_{AB} = (0)$, where $H = (h_{ij})$, so that $[N_H, M_{AB}] \in D_{r-k-t}[N, M]$. To show that $N_H \cap M_{AB} = (0)$; let $y \in N_H \cap M_{AB}$. Then y takes the form

$$y = \sum_{i=1}^{r-k-t} \alpha_i y_i = \sum_{i=1}^{k} \beta_i v_i + \sum_{i=k+1}^{n-r+k} \gamma_i u_i + \sum_{i=1}^{t} \delta_i z_i';$$

from which we can obtain that $\gamma_i = 0$ for $i = k + 1, k + 2, \ldots, n - r + k$, and that

$$-\sum_{i=1}^{r-k-t} \alpha_i y_i + \sum_{i=1}^{k} \beta_i v_i + \sum_{i=1}^{t} \delta_i z_i' = 0.$$

By computation, we can see that $\delta_1 = \delta_2 = \cdots = \delta_t = 0$, $\alpha_1 = \alpha_2 = \cdots = \alpha_{r-k-t} = 0$ and hence $\beta_1 = \beta_2 = \cdots = \beta_k = 0$. Thus $y = 0$ and $N_H \cap M_{AB} = (0)$. We can prove that $N_H + M_{AB} = V$. Finally, by observing $(h_{ij}) = H$ in $F(t, r - k - t)$ in the expression $z_i' = z_i + \sum_{j=1}^{r-k-t} h_{ij} x_j$ (and by a similar argument of (iii) if it is necessary) we can infer that the number m_3 of all N_H of $[N_H, M_{AB}]$ in $D_{r-k-t}[N, M]$ is equal to $q^{t(r-k-t)} = |F(t, r - k - t)|$.

(v) We conclude that

$$m = |D_{r-k-t}[N, M]| = m_1 m_2 m_3 = \begin{bmatrix} r - k \\ r - k - t \end{bmatrix} q^{(k+t)(r-k-t)}.$$

(3) (i) If $M + N \neq V$, then it is clear that $\dim(M \cap N) > k$. Let $\dim(M \cap N) = k + m$ $(m > 0)$. Let

$$M \cap N = [v_1, v_2, \ldots, v_{k+m}],$$

$$M = [v_1, v_2, \ldots, v_{k+m}, v_{k+m+1}, \ldots, v_r]$$

and

$$N = [v_1, v_2, \ldots, v_{k+m}, u_{k+m+1}, u_{k+m+2}, \ldots, u_{n-r+k}].$$

If $[N_1, M_1] \in D_{r-k-t}[N, M]$, then it is not difficult to infer that $r - k - t = \dim M_1 \leqslant r - (k + m)$, from which we have that $m \leqslant t$. Thus if $m > t$, then $D_{r-k-t}[N, M] = \emptyset$.

(ii) Assume that $m \leqslant t$. If we choose $(r - k - t)$ linearly independent elements $\{x_1, x_2, \ldots, x_{r-k-t}\}$ from the vector space $[v_{k+m+1}, v_{k+m+2}, \ldots, v_r]$, then we can show that

$$M_1 = \left[x_1 + \sum_{j=1}^{k+m} c_{1j}v_j,\ x_2 + \sum_{j=1}^{k+m} c_{2j}v_j, \ldots, x_{r-k-t} + \sum_{j=1}^{k+m} c_{r-k-t\,j}v_j\right] \subset M,$$

$$\dim M_1 = r - k - t$$

and that there is N_1 such that $N \subset N_1$ and $M_1 \oplus N_1 = V$; hence $[N_1, M_1] \in D_{r-k-t}[N, M]$ for such a space N_1. With this and consideration of the arguments in the proof of (2), we can infer that the number of all M' such that $[N', M'] \in D_{r-k-t}[N, M]$, for a fixed N', is equal to

$$\begin{bmatrix} r - k - m \\ r - k - t \end{bmatrix} q^{(r-k-t)(k+m)}.$$

(iii) Now consider N' of $[N', M']$ in $D_{r-k-t}[N, M]$, for a fixed N'. We shall show that the number of all such N' is equal to $q^{t(r-k-t)}$, so that

$$|D_{r-k-t}[N, M]| = \begin{bmatrix} r - k - m \\ r - k - t \end{bmatrix} q^{(k+m+t)(r-k-t)}.$$

Let

$$\{v_j : j = 1, 2, \ldots, r\} \cup \{u_j : j = k + m + 1, k + m + 2, \ldots, n - r + k\}$$
$$\cup \{e_j : j = 1, 2, \ldots, m\}$$

be a basis for the space V. Let $\{x_1, x_2, \ldots, x_{r-k-t}\}$ be a linearly independent vectors in $[v_{k+m+1}, v_{k+m+2}, \ldots, v_r]$. Let

$$M_1 = [x_1, x_2, \ldots, x_{r-k-t}]$$

and

$$[x_1, x_2, \ldots, x_{r-k-t}, x_{r-k-t+1}, x_{r-k-t+2}, \ldots, x_{r-k-m}] = [v_{k+m+1}, v_{k+m+2}, \ldots, v_r].$$

Let $B = (c_{ij})$ be a $t \times (r - k - t)$ matrix over F. Let

$$y_1 = e_1 + \sum_{j=1}^{r-k-t} c_{1j}x_j,$$

$$y_2 = e_2 + \sum_{j=1}^{r-k-t} c_{2j}x_j, \ldots,$$

$$y_m = e_m + \sum_{j=1}^{r-k-t} c_{mj}x_j,$$

$$y_{r-k-t+1} = x_{r-k-t+1} + \sum_{j=1}^{r-k-t} c_{r-k-t+1\ j}x_j,$$

$$y_{r-k-t+2} = x_{r-k-t+2} + \sum_{j=1}^{r-k-t} c_{r-k-t+2\ j}x_j, \ldots,$$

$$y_{r-k-m} = x_{r-k-m} + \sum_{j=1}^{r-k-t} c_{r-k-m\ j}x_j.$$

Then letting

$$N_1 = [y_1, y_2, \ldots, y_m, y_{r-k-t+1}, y_{r-k-t+2}, \ldots, y_{r-k-m}, v_1, v_2, \ldots, v_{k+m},$$
$$u_{k+m+1}, u_{k+m+2}, \ldots, u_{n-r+k}],$$

we can see that $[N_1, M_1] \in D_{r-k-t}[N, M]$. Thus we can infer that the number of all N' such that $[N', M'] \in D_{r-k-t}[N, M]$ for a fixed M' is equal to $q^{t(r-k-t)}$. We conclude that

$$|D_{r-k-t}[N, M]| = \begin{bmatrix} r-k-m \\ r-k-t \end{bmatrix} q^{(k+m+t)(r-k-t)}.$$

This proves the Lemma 5.

To consider all possible cases $M + N = V$, $M + N \neq V$, $M \cap N = (0)$ and $M \cap N \neq (0)$ we shall have the following lemma.

LEMMA 6. *Assume that $M \in \pi_r(V)$, $N \in \pi_s(V)$ and $M \cap N = (0)$. Let $1 \leqslant i \leqslant r$.*

(1) *If $r, s \geqslant 1$, then*

$$|D_i[N, M]| = \begin{bmatrix} r \\ i \end{bmatrix} q^{i(n-s-i)}.$$

(2) *If* $r = 0$, *then* $M = (0)$ *and* $|D_i[N, (0)]| = 0$.
(3) *If* $s = 0$, *then* $N = (0)$ *and*

$$|D_i[(0), M]| = \begin{bmatrix} r \\ i \end{bmatrix} q^{i(n-i)}.$$

The proof of the lemma is technically the same as the proof of Lemma 5 and we shall omit the proof of this lemma.

3. THEOREM

To state the main theorem we translate Lemma 6 into the following version.

LEMMA 7. *Let V be an n-dimensional vector space over a finite field F with q elements. Let M and N be two subspaces of V of dimensions r and s, respectively, with $M \cap N = (0)$. Let i be a positive integer less than or equal to r.*

(1) *If* $r, s \geqslant 1$, *then the number m of all idempotent linear transformations T of V of rank i such that $R(T) \subset M$ and $N \subset N(T)$ is equal to*

$$m = \begin{bmatrix} r \\ i \end{bmatrix} q^{i(n-s-i)}.$$

(2) *If* $M = (0)$, *then* $m = 0$.
(3) *If* $N = (0)$, *then*

$$m = \begin{bmatrix} r \\ i \end{bmatrix} q^{i(n-i)}.$$

The proof of Lemma 7 follows from Theorem 1 and Lemma 6. The following theorem summarizes Lemmas 4, 5, and 6 in terms of idempotent linear transformations of V.

THEOREM. *Let M and N be two subspaces of a finite dimensional vector space V over a finite field F. Then the number m of all idempotent linear transformations T of V of rank i such that $R(T) \subset M$ and $N \subset N(T)$ is equal to*

$$m = |D_i[N, M]|$$
$$= \begin{bmatrix} \dim M - \dim(M \cap N) \\ i \end{bmatrix} |F|^{i[\dim V + \dim(M \cap N) - \dim N - i]}.$$

Proof. By Theorem 1, each $[N', M']$ in $D_i[N, M]$ uniquely determines an idempotent linear transformation T of V with range$(T) = M'$ and kernel$(T) = N'$. Therefore it is easy to see that the number m is equal to $|D_i[N, M]|$ in view of Lemma 7 and Theorem 1. Thus all we need is to check that

$$\begin{bmatrix} \dim M - \dim(M \cap N) \\ i \end{bmatrix} |F|^{i[\dim V + \dim(M \cap N) - \dim N - i]}$$

is equal to the number in each item of Lemmas 4, 5, and 6.

Consider Lemma 4. If $i = 0$, then we can see that

$$\begin{bmatrix} \dim M \\ 0 \end{bmatrix} |F|^0 = 1$$

because of the definition of

$$\begin{bmatrix} r \\ 0 \end{bmatrix} = 1$$

in Lemma 3. Hence, this theorem covers Lemma 4. We now check four items of Lemma 7 instead of Lemma 6. The condition $\dim(M \cap N) = 0$ is imposed in this case. For Lemma 7(1), we can see that

$$\begin{bmatrix} \dim M - \dim(M \cap N) \\ i \end{bmatrix} |F|^{i(\dim V + \dim(M \cap N) - \dim N - i)} = \begin{bmatrix} r \\ i \end{bmatrix} q^{i(n-s-i)},$$

which is the number in Lemma 7(1). If $\dim M = 0$, then the expression m in the theorem reduces to

$$\begin{bmatrix} 0 \\ i \end{bmatrix} q^{i(n-s-i)} = 0$$

if $i \geqslant 1$, which checking Lemma 7(2). For Lemma 7(3), if $\dim N = 0$, then the expression in the theorem reduces to

$$\begin{bmatrix} r \\ i \end{bmatrix} q^{i(n-i)}.$$

Hence, this theorem covers Lemma 7 and hence it includes Lemma 6. We now consider Lemma 5. It is an agreement that

$$\begin{bmatrix} r \\ r+i \end{bmatrix} = 0 \quad \text{for} \quad i \geqslant 1.$$

Hence Lemma 5(1) is included in the theorem. For Lemma 5(2), if $M + N = V$, then $\dim(M \cap N) = k$. Let $i = r - k - t$. We have that $\dim V + \dim(M \cap N) - \dim N - i = n + k - (n - r + k) - i = r - i$ and $|F| = q$. Thus we can see that

$$\begin{bmatrix} r-k \\ r-k-t \end{bmatrix} q^{(k+t)(r-k-t)}$$

$$= \begin{bmatrix} \dim M - \dim(M \cap N) \\ i \end{bmatrix} |F|^{i(\dim V + \dim(M \cap N) - \dim N - i)},$$

since $k + t = r - i$. Now consider Lemma 5(3). Let $i = r - k - t$ and let $\dim(M \cap N) = k + m$. Then $\dim V + \dim(M \cap N) - \dim N - i = m + k + t$. This proves the theorem.

4. APPLICATION

As an application of our results, we take a simple problem [2, Exercise 6(a)]:

Can it happen that a nontrivial subspace of a vector space V has a unique complement? Of course, the answer to this question is "No." If M is a subspace of the vector space V of dimension $r > 1$, then, taking $N = (0)$, by the theorem, we have that $|D_r[(0), M]| = |F|^{r(\dim V - r)}$, as the number of all complementary nontrivial subspaces N' of M in V for a fixed M.

REMARK. By Theorem 1, each $[N, M]$ uniquely determines an idempotent linear transformation T of V with $\text{range}(T) = M$ and $\text{kernel}(T) = N$. Thus the problem is equivalent to:

Can it happen that a nontrivial subspace M of a vector space V has a unique idempotent T with $\text{range}(T) = M$? If $\dim M = r$, then the number of all idempotent linear transformations T of V with $\text{range}(T) = M$ is equal to $|D_r[(0), M]| = |F|^{r(\dim V - r)}$ by the theorem.

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